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OBLIQUE INCIDENCE OF SURFACE WAVES ON AN ELASTIC PLATE

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Oblique incidence of small-amplitude waves on an elastic plate floating on a free liquid surface is studied. The reflection and transmission factors of the waves and the vertical displacements and strains of the plate are determined. It is shown that all these characteristics depend greatly on the incidence angle and frequency of the wave and the width of the plate. Approximate solutions for the reflection and transmission factors are obtained.

The behavior of a thin elastic plate floating on a disturbed water surface has been actively investigated in connection with studies of flexural-gravity waves in a liquid with an ice floe (review [1, 2]). In addition, interest in this problem has increased recently because of the design of buoyant island and platforms for various uses [3]. Usually, these man-made structures have a rectangular shape with a large (approximately 10 : 1) length-to-width ratio. It can be assumed that away from the corner points, the behavior of the plate is approximately described by the solution of the problem of surface waves incident on an elastic plate of infinite length and constant width. Let the liquid be ideal and incompressible, have constant depth, and be infinite along the horizontal direction, and the liquid flow be potential. The external disturbance is considered planar and regular, and the amplitudes of the surface waves and the flexural vibrations of the elastic plate are considered small. The particular cases of the problem of oblique incidence of waves on the edge of a semi-infinite plate and normal incidence on a plate of finite width are examined, respectively, by Fox and Squire [4] and Meylan and Squire [5].

Let an elastic, infinitely long plate of width L and thickness h float on the free surface of a liquid layer of depth H. The settling of the plate is assumed to be negligible. From the region of the free surface of the basin, a progressive wave with frequency ω is obliquely incident on the left edge of the plate. The coordinate system is chosen so that the coordinate origin is located at the bottom of the basin under the left edge of the plate, the horizontal x axis is perpendicular to the edges of the plate, the y axis is parallel to them, and the z axis is directed vertically upward.

The incident wave propagates at angle θ to the x axis and is defined by the velocity potential

$$\Phi_0(\boldsymbol{x},t) = \varphi_0(x,z)\exp\left[i(\omega t - \beta y)\right],$$

where $\varphi_0 = iag\cosh(k_0 z)/(\omega\cosh(k_0 H))\exp(-i\alpha x)$, $(\alpha,\beta) = k_0(\cos\theta,\sin\theta)$, $\boldsymbol{x} = (x,y,z)$, a is the wave amplitude, g is the acceleration due to gravity, and the wavenumber k_0 is a positive real root of the equation

$$\omega^2 = gk_0\tanh(k_0 H). \tag{1}$$

Here and below, in all the expressions containing the factor $\exp(i\omega t)$, only the real part has a physical meaning.

Steady-state waves are considered, and because of the infinite length of the elastic plate, the velocity potential of the disturbed liquid flow is sought in the form

$$\Phi(\boldsymbol{x},t) = \varphi(x,z)\exp\left[i(\omega t - \beta y)\right]. \tag{2}$$

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To determine $\varphi(x, z)$, it is necessary to solve the equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} - \beta^2 \varphi = 0 \qquad (3)$$

with the boundary conditions

$$\frac{\partial \varphi}{\partial z} - \frac{\omega^2}{g} \varphi = 0 \qquad (x < 0, \quad x > L, \quad z = H); \qquad (4)$$

$$\Big[D\Big(\frac{\partial^2}{\partial x^2} - \beta^2\Big)^2 + 1 - \mu\omega^2\Big] \frac{\partial \varphi}{\partial z} - \frac{\omega^2}{g} \varphi = 0 \qquad (0 < x < L, \quad z = H); \qquad (5)$$

$$\frac{\partial \varphi}{\partial z} = 0 \qquad (z = 0), \qquad (6)$$

where $D = Eh^3/(12\rho g(1 - \nu^2))$ and $\mu = \rho_1 h/(\rho g)$. Here E, ρ_1, and ν is the modulus of normal elasticity, density, and Poisson's ratio of the plate and ρ is the water density. It is assumed that the plate contacts water at all points and at all times. At the edges of the plate, the free-edge conditions are satisfied, which imply that the flexural moment and the cutting force are equal to zero:

$$\frac{\partial}{\partial z}\Big(\frac{\partial^2 \varphi}{\partial x^2} - \nu\beta^2\varphi\Big) = \frac{\partial^2}{\partial x\, \partial z}\Big[\frac{\partial^2 \varphi}{\partial x^2} - (2 - \nu)\beta^2 \varphi\Big] = 0 \quad (x = 0^+, \ x = L^-, \ z = H). \qquad (7)$$

To solve problem (3)–(7), we use the conjugation method of [6], in which the region S occupied by the liquid is divided into three parts: S_1 $(-\infty < x < 0)$, S_2 $(0 < x < L)$, and S_3 $(L < x < \infty)$; in each of them $\varphi(x, z)$ is denoted by $\varphi_j(x, z)$ $(j = \overline{1, 3})$. Further, we convert to dimensionless variables by choosing the basin depth H as the length scale and $\sqrt{H/g}$ as the time scale.

The functions φ_j are sought in the form of expansion in the eigenfunctions of the corresponding boundary-value problems:

$$\varphi_1 = [E_0 \exp(-i\alpha x) + A_0 \exp(i\alpha x)] Y_0(k_0, z) + \sum_{n=1}^{\infty} A_n \exp(\alpha_n x) Y_1(k_n, z); \qquad (8)$$

$$\varphi_2 = [B_0 \exp(-iq_0 x) + C_0 \exp(iq_0 x)] Y_0(r_0, z) + \sum_{m=1}^{4} G_m \exp(s_m x) \cos(p_m z)$$

$$+ \sum_{n=1}^{\infty} [B_n \exp(-q_n x) + C_n \exp(q_n x)] Y_1(r_n, z); \qquad (9)$$

$$\varphi_3 = F_0 \exp(-i\alpha x) Y_0(k_0, z) + \sum_{n=1}^{\infty} F_n \exp(-\alpha_n x) Y_1(k_n, z). \qquad (10)$$

Here $E_0 = ia\sqrt{\Lambda_0(k_0)}/(\omega \cosh k_0)$, k_n $(n = 1, 2, \ldots)$ are real roots of the equation

$$\omega^2 = -k_n \tan k_n, \qquad (11)$$

$\alpha_n = \sqrt{k_n^2 + \beta^2}$, r_0 is a real root of the equation

$$\omega^2 = \frac{(1 + \delta r^4) r \tanh r}{1 + \gamma r \tanh r}, \qquad (12)$$

$$q_0 = \sqrt{r_0^2 - \beta^2}, \qquad (13)$$

$\delta = D/H^4$, and $\gamma = \mu g/H$. Equation (12) has an infinite number of purely imaginary roots $\pm ir_n$ $(n = 1, 2, \ldots)$, $q_n = \sqrt{r_n^2 + \beta^2}$ and four complex roots $\pm\sigma \pm i\lambda$ ($\sigma > 0$ and $\lambda > 0$). The values of p_m are equal to $\pm\lambda \mp i\sigma$; $s_m = \sqrt{p_m^2 + \beta^2}$. We enumerate the values of s_m as follows: $s_{1,2} = c \pm id$, $s_{3,4} = -c \pm id$ ($c > 0$ and $d > 0$).

The functions Y_0 and Y_n $(n = 1, 2, \ldots)$ have the form

$$Y_0(\xi, z) = \frac{\cosh(\xi z)}{\sqrt{\Lambda_0(\xi)}}, \qquad \Lambda_0(\xi) = \int_0^1 \cosh^2(\xi z)\, dz = \frac{1}{2} + \frac{\sinh(2\xi)}{4\xi},$$

$$Y_1(\xi, z) = \frac{\cos(\xi z)}{\sqrt{\Lambda_1(\xi)}}, \qquad \Lambda_1(\xi) = \int_0^1 \cos^2(\xi z)\, dz = \frac{1}{2} + \frac{\sin(2\xi)}{4\xi}.$$

The properties of the eigenvalues and eigenfunctions are extensively studied in [4, 7]; we indicate their basic properties briefly. In the regions S_1 and S_3, the eigenfunctions are orthogonal and represent a complete system for the potential satisfying Eq. (3) and boundary conditions (4) and (6). In the dimensionless variables according to Eq. (1), $k_0 \approx \omega$ as $\omega \to 0$ and $k_0 \approx \omega^2$ as $\omega \to \infty$. The roots k_n of Eq. (11) are such that $(n - 1/2)\pi < k_n < n\pi$ and $k_n \to n\pi$ for large n. In the region S_2, the eigenfunctions are nonorthogonal, but they also represent a complete system. The real root of Eq. (12) is $r_0 \approx \omega$ as $\omega \to 0$, and $r_0 \approx \omega^{1/2}(\gamma/\delta)^{1/4}$ as $\omega \to \infty$. The roots r_n obey the inequalities $(n - 1/2)\pi < r_n < (n + 1/2)\pi$ and $r_n \to n\pi$ for large n. One of the complex roots of Eq. (12) for $\omega = 0$ is equal to $\sigma + i\lambda = (1 + i)2^{-1/2}\delta^{-1/4}$. The modes related to k_n and r_n are called edge modes and the modes determined by the complex roots p_m $(m = \overline{1,4})$ and due to the flexural rigidity of ice are called growing or damped progressive waves, depending on the sign of $\mathrm{Re}(s_m)$. In the numbering introduced above, the modes determined by s_1 and s_2 are growing and the modes determined by s_3 and s_4 are damped. The mode related to r_0 is a progressive wave for real values of q_0 in (13). However, for $r_0 < \beta$, the value of q_0 becomes imaginary, and this corresponds to the boundary mode. The value of the angle $\theta = \theta_0$, where

$$\theta_0 = \arcsin(r_0/k_0), \tag{14}$$

is called critical since complete reflection of incident progressive waves from the plate takes place. A similar phenomenon is also observed for oblique incidence of surface waves on a rectangular base trench [8].

By virtue of the liquid-flow continuity in the region S, matching conditions for the pressure and horizontal velocity are imposed on the boundaries of the regions S_j, and, hence,

$$\varphi_1 = \varphi_2, \qquad \frac{\partial \varphi_1}{\partial x} = \frac{\partial \varphi_2}{\partial x} \qquad (x = 0, \quad 0 \leqslant z \leqslant H); \tag{15}$$

$$\varphi_2 = \varphi_3, \qquad \frac{\partial \varphi_2}{\partial x} = \frac{\partial \varphi_3}{\partial x} \qquad (x = L, \quad 0 \leqslant z \leqslant H). \tag{16}$$

Using the reduction method, we replace the infinite series in (8)–(10) by finite sums with N terms. The matching conditions (15) and (16) are satisfied in the integral sense, i.e., they are successively multiplied by the functions $Y_0(k_0, z)$, $Y_0(r_0, z)$, $Y_1(k_n, z)$, and $Y_1(r_n, z)$ $(n = 1, 2, \ldots)$ and integrated in the interval $0 \leqslant z \leqslant H$. The constants A_0, A_n, F_0, and F_n are conveniently expressed in terms of the remaining unknown complex constants, and as a result, the problem is reduced to solving $2N + 6$ linear equations, which is implemented numerically.

In studies of wave diffraction on a plate, an approximation is frequently used that ignores edge waves, i.e., the finite sums in the representations (8)–(10). This problem is much easier to solve since it is reduced to a system of only six linear equations.

In the limit $L \to \infty$, the problem considered corresponds to oblique incidence of surface waves on the edge of a semi-infinite plate. In this case, the region S is divided into two parts: S_1 $(x < 0)$ and S_2 $(x > 0)$. The representation for φ_1 still has the form (8), and for φ_2 in (9), terms with the coefficients C_0, G_1, G_2, and C_n should be omitted. On the boundary between the regions S_1 and S_2, the matching conditions (15) must be satisfied. This problem is also solved by the method of integral gluing, and the problem is reduced to a system of $3 + N$ linear equations. To derive an approximate solution ignoring edge waves, one should solve a system of three linear equations.

Calculating all the unknown constants in (8)–(10), one can determine the wave flow of the liquid and the strain of the plate. For a plate of finite width, the reflection factor R and the transmission factor T, which characterize, respectively, the surface progressive wave reflected from the plate and the wave transmitted through it, are

$$R = A_0/E_0, \qquad T = F_0/E_0 \tag{17}$$

and obey the energy relation

$$|R|^2 + |T|^2 = 1. \tag{18}$$

For a semi-infinite plate, we introduce the notation R_1 for the reflection factor, which is determined similarly to R in (17), and T_1 for the transmission of an undamped progressive flexural-gravity wave in the plate:

$$T_1 = \frac{r_0 B_0 \sinh r_0}{k_0 E_0 \sinh k_0} \sqrt{\frac{\Lambda_0(k_0)}{\Lambda_0(r_0)}}.$$

Fox and Squire [4] give the energy relation

$$|R_1|^2 + Q|T_1|^2 = 1, \tag{19}$$

where

$$Q = \frac{\operatorname{Re}(q_0) k_0^2 \sinh(2k_0)[2r_0(\delta r_0^4 + 1 - \gamma\omega^2) + \sinh(2r_0)(5\delta r_0^4 + 1 - \gamma\omega^2)]}{\operatorname{Re}(\alpha) r_0^2 \sinh(2r_0)[2k_0 + \sinh(2k_0)]}.$$

Using a method similar to the one for solving optical problems, Meylan and Squire [9] derive formulas relating the reflection and transmission factors for plates of finite and infinite width for normally incident waves in the approximation of single dispersion. Extending these formulas to the case of obliquely incident waves, we obtain

$$R = R_1 - \frac{R_1^* T_1^2 (1 - |R_1|^2)}{(T_1^*)^2 \exp(2iq_0 L) - T_1^2 (R_1^*)^2}, \quad T = \frac{|T_1|^2 (1 - |R_1|^2) \exp[i(\alpha + q_0)L]}{(T_1^*)^2 \exp(2iq_0 L) - T_1^2 (R_1^*)^2}, \tag{20}$$

where the asterisk denotes complex conjugation. As pointed out in [9], this approximation is valid only when the width of the plate is great compared to the length of a flexural-gravity wave $l_r = 2\pi/r_0$. As for a rigid plate [6], there is a discrete spectrum of frequencies for which $R = 0$ (the so-called spectral windows). An increase in the plate length at fixed remaining parameters leads to complete transmission of an incident wave with interval $\Delta L = \pi/q_0$ for a real value of q_0.

Similarly to (2), the vertical displacements of the plate $\eta(x, y, t)$ can be written as

$$\eta(x, y, t) = \operatorname{Re}\{\zeta(x) \exp[i(\omega t - \beta y)]\}.$$

Using the relation

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z}\Big|_{z=H},$$

we obtain

$$zeta(x) = \frac{1}{i\omega} \frac{\partial \varphi_2(x, z)}{\partial z}\Big|_{z=H}.$$

The strain tensor ε is the matrix

$$\varepsilon = -\frac{h}{2} \begin{Vmatrix} \dfrac{\partial^2 \eta}{\partial x^2} & \dfrac{\partial^2 \eta}{\partial x \, \partial y} \\ \dfrac{\partial^2 \eta}{\partial x \, \partial y} & \dfrac{\partial^2 \eta}{\partial y^2} \end{Vmatrix},$$

for which the eigenvalues corresponding to the principal strains can be determined at each time. The maximum strain ε_m is calculated as the greatest in absolute value of the two principal strains with variation in ωt from

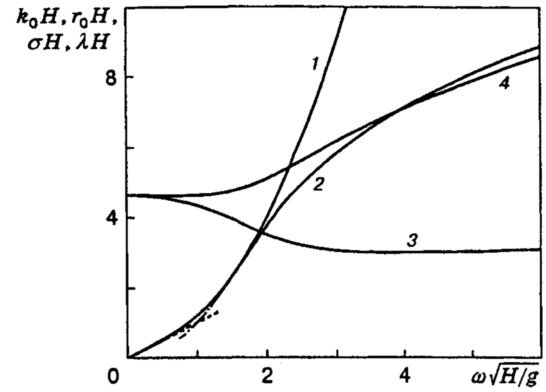

Fig. 1

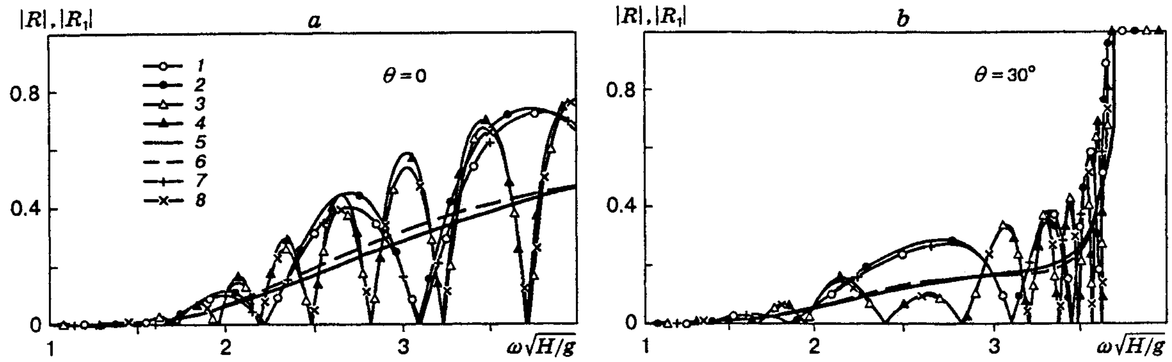

Fig. 2

0 to 2π. Along with the maximum strain, the normal strain ε_n was calculated to be equal to $\frac{h}{2}\left|\frac{\partial^2\eta}{\partial x^2}\right|$. At a normally incident wave, $\varepsilon_n = \varepsilon_m$.

The numerical calculations are performed for parameters of an ice floe that are most frequently used in the literature cited (below, we revert to dimensional variables): $E = 6 \cdot 10^9$ Pa, $H = 100$ m, $\rho = 1025$ kg/m^3, $h = a = 1$ m, $\rho_1 = 922.5$ kg/m^3, and $\nu = 0.3$. For these values, $\delta = 5.47 \cdot 10^{-4}$ and $\gamma = 9 \cdot 10^{-3}$.

Figure 1 shows the kinematic characteristics of progressive waves: curve 1 shows the behavior of the wavenumber k_0H for a gravity wave, curve 2 shows the behavior of the wavenumber r_0H for a flexural-gravity wave, and curves 3 and 4 are the values of σH and λH, respectively. For a specified value of δ, we have $\sigma H = \lambda H \approx 4.62$ for $\omega = 0$. The dashed curve represents the dependence $\omega = k_0\sqrt{gH}$ for long waves, and the dot-and-dashed curve is the high-frequency approximation $\omega = \sqrt{gk_0}$. It is obvious that the behavior of k_0 is adequately described by these two approximations. The high-frequency behavior of r_0 is not shown because, by virtue of the smallness of the values of δ and γ, it takes place only for $\omega \gg 1$.

Figure 2a and b shows the reflection factors at $\theta = 0$ and $\theta = 30°$, respectively (for $\omega\sqrt{H/g} < 1$ in these cases, $|R|$ and $|R_1|$ are smaller than 0.01). Curves 1, 2, and 7 correspond to a plate of width $L/H = 2$, curves 3, 4, and 8 correspond to a plate of width $L/H = 5$, and curves 5, and 6 correspond to a semi-infinite plate ($L \to \infty$). Curves 1, 3, and 5 represent a numerical solution that takes into account $N = 20$ boundary modes, and curves 2, 4, and 6 represent an approximate solution without edge modes. With allowance for the edge modes, the error of satisfaction of the energy relations (18) and (19) for plates of finite and infinite width, respectively, do not exceed 1%. It is obvious that the approximate solution that ignores the edge waves adequately describes the behavior of the reflection factor, and also, according to (18) and (19), the transmission factor for rather long waves. The approximate values of (20) shown by curves 7 and 8 are in good agreement with the complete solution (curves 1, and 3). An interesting feature of obliquely incident waves is that the oscillation of the reflection factor is enhanced as the frequency corresponding to the critical

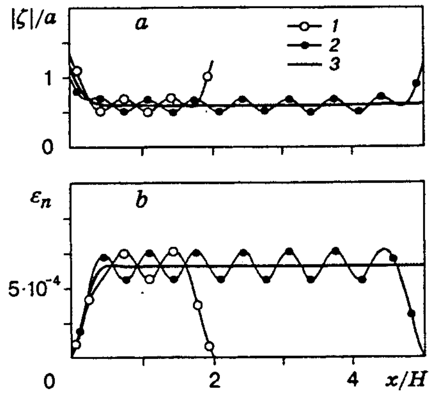

Fig. 3

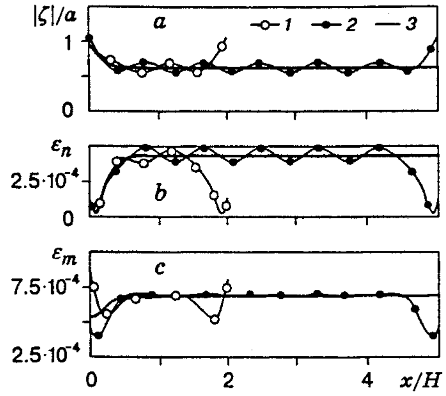

Fig. 4

angle is approached [according to (14) $\theta_0 = 30°$ as $\omega\sqrt{H/g} \approx 3.67$].

The strain characteristics of the plate are shown in Figs. 3 and 4 for normally and obliquely ($\theta = 30°$) incident waves, respectively, at $\omega\sqrt{H/g} = 2.4$ (incident-wave length $l_k = 2\pi/k_0 \approx 109$ m and wave period $\tau = 2\pi/\omega \approx 8.4$ sec). Figure 3a and b and Fig. 4a and b show the amplitudes of vertical displacements $|\zeta|/a$ and normal strains ε_n along the plate. Curves 1–3 correspond to $L/H = 2$, 5, and ∞. For normal incidence of waves on a plate of finite width, the amplitudes of vertical displacements and strains exhibit nonmonotonic behavior along the plate, and maximum vertical displacements are reached at the ends of the plate. Similar behavior of the indicated characteristics is noted in [3]. Calculations for the parameters indicated in [3] showed satisfactory agreement between the theoretical and experimental results. The values of ε_n at the plate ends are equal to zero by virtue of boundary conditions (7). In addition, Fig. 4c shows the distribution of the maximum strains ε_m for oblique incidence.

An interesting feature of the oblique incidence of waves on a plate of finite width in the neighborhood of the critical angle is that maximum vertical displacements can occur in its middle part and not at the plate ends.

For complete reflection of an incident wave, elastic deformations of the plate are observed only at the leading edge, and their behavior does not depend on the width of the plate. A semi-infinite plate is examined in detail in [4].

The accuracy of the calculations described here was checked by increasing the number of edge modes successively. For the purposes of the present work, it is sufficient to set $N = 20$.

The studies performed can also be used to determine the spectral characteristics of hydroelastic deformations of a plate with specification of a concrete spectrum of wind waves similarly to [5, 10].

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